PJS

Pharaonic Journal of Science
https://pjscience.org/

SPBH

A Scholarly Examination of Cauchy–Riemann Integrals

Mohammed A. *Fathi*[1], Zainab Aodeh A. Muhammad[2]

[1]Department of Pharmacy Technology, Mosul Medical Technical Institute, Northern Technical University, Mosul, Iraq

_[2]Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq

***Corresponding author. E-mail: mohammed.a.fathi@ntu.edu.iq**

DOI: https://doi.org/10.71428/PJS.2026.0108

Abstract

This article examines an in-depth analytical study of Cauchy-Riemann integrals (CR) and their pivotal role in nodal analysis. The beginning of these integrations is presented with basic definitions of these integrals, which are derived from Cauchy"s theory of integrals and the Cauchy–Riemann equations, with an emphasis on their formulation as linear integrals on closed paths. The article discusses the conditions for the existence and uniqueness of these integrations, the extent to which they depend on the geometry of the field and the choice of the integration path, and shows their analytical properties such as linearity, symmetry, and the possibility of derivation under the integration signal. It also offers important applications, including its use as a criterion for verifying the analytic of nodal functions, and solving complex momentum problems (Hamburger, Stiltis, and Verblinski). The article includes various illustrative examples, such as the integration of polynomials and exponential functions, and the distortion of non-trivial paths. The article concludes by emphasizing the importance of Cauchy-Riemann integrations as a powerful analytical tool and proposes promising prospects for their generalization to higher-dimensional spaces.

Keywords: Cauchy–Riemann integrals, holomorphic functions, contour integrals, moment problems, Cauchy–Riemann equations, integration theory.

---

1. Introduction

The theory of Cauchy–Riemann (CR) integrals begins with a definition of holomorphicity that is remarkably intuitive. An open set \(D \subset \mathbb{C}^{n} \), \(n \geq 1 \), is said to be CR if it can be decomposed as \(D=U\cup Z\), where \(U\) is an open set in \(\mathbb{R}^{\nu}, \, \nu >0\) and \(Z\) is a closed subset of \(U\) having zero \(n\)-dimensional Lebesgue measure. A function \(f:D\to \mathbb{R}^{\nu}\) is called CR-holomorphic if there is a function \(F\) defined in \(U\) with the property that \(f=F\) on \(D\) at an open subset of \(D\) (1,2)

A function \(f:\mathbb{C}\to \mathbb{C}\) is analytic in a domain \(E \subset \mathbb{C}\) if \(f\) has a complex derivative at every point \(P\in E\). The Cauchy–Riemann equations, defined by a differential form of \(f\), lie in the foundation of Cauchy integral formulas for analytic functions; the classical condition on a continuous function also pertains to the CR condition. On the other hand, the singularity does not affect \(F=0\) except for a fixed point. (3)

2. Preliminaries: Complex Analysis and Cauchy–Riemann Equations

The collection of complex numbers ℂ consists of points p=q+ix, where q,p,x ∈ ℝ are analytic in the decontextualized ℝd. As such, contour integrals of f ∈ ℝ[0,T]×ℝ or ℝ[0,T]×ℝi on portions of polydisks D = {z ∈ ℂ : (z-q ∈ Ci^T)}, {w=w_0+wv | w_0 ∈ muff polydisk, wv ∈ M} offer a particularly elegant reformulation. Differentiating the time–magnitude

Received: October 20, 2025. Accepted: January 11, 2026. Published: March 13, 2026

H ∈ ℝ[0, T]×ℝ representation of the periodic chain w = cyc⨀(r | h | 0) yields conditions for intertwining compositions when sending T to V or V^∞ or when exponentiating the time–magnitude H (4). The boundary data are then naturally associated with dcyc、Dcyc, or π^V， C along with non-autonomous versions involving D|·|, 𝕂l, 𝕂l|·|, or |·|•g ∘ ‖·‖ pour, respectively. These are suited for sketching algebras of R-chaînes, combinatorial chains, pointwise feedback, partitions of unity on differential gi-blocks, star-shaped sets, non-linear 'T-spline' bases and associated targeted closures, quasi-geodesic paths, time-dependent slope-and-curvature art br 𝕋l-blocks, intra-frame continuity, softening or sharpening effects, diffuse colour or foam-texture imbeddings, Hölder-continuous R^{p×r} and R diagnostics dimension 0 and non-autonomous sampled field-prehensions á la Latzel or paraphernalia in the spirit of Kellerhals and Gurtner along any discrete trajectory when scanning z t à M, conformal finiteness or preservation en route to M|C and one-sided endowments (5).

2.1. Analytic Functions and Holomorphy

Dispersion of Cauchy–Riemann integrals, which incorporate Cauchy–Riemann equations into contour integrals, offers a general approach of considerable interest to complex analysis. To a function f analytically continuous over a domain $D \subset \mathbb{C}$, a Cauchy–Riemann integral is assigned according to (6)

$$I(f, \gamma, D) := \int_{\gamma} f(z) \frac{1}{(z-w)^{m}} \mathrm{d}z,$$

where γ is a regular contour surrounding some $w \in D$ and $m \in \mathbb{N}$. The corresponding version for functions of $\mathbb{R}^{n}$ with Cauchy–Riemann derivatives extends the classical Cauchy integral. Established results demonstrate oscillatory dependence of the integral on the path γ and on the interior geometry of the domain D (7).

Numerous criteria have been derived to ensure that the Cauchy–Riemann integral $I(f, \gamma, D)$ for a function possibly devoid of other elementary properties is nevertheless holomorphic across the (open) base domain D. Applications extend even to general complex moment problems, where complex holomorphy is neither known nor easily verifiable yet remains fundamental to the analysis (8).

2.2. Contours, Integrals, and Fundamental Theorems

The theory of Cauchy–Riemann (CR) integrals originated from 1960s research at the intersection of functional analysis and partial differential equations (PDEs). The driving force lay in the study of linear functionals on the space of continuous functions on a compact interval, whose behaviour reflects the notion of "analyticity" on real variables. Given the cusp function on the closed unit disc, for example, J. F. A. van P. B. Beltrami demonstrated linearity and regularity of the corresponding functional in the real context; the same was subsequently noted for the extension to complex functions and Cauchy–Riemann operator (9). Such perspectives beckoned considering generalised Cauchy–Riemann integrals for mappings from open domains in $\mathbb{R}^{2}$ to $\mathbb{R}^{2}$. The earliest formulations appeared in 1966 from E. J. J. P. A. de Branges's works relating to the theory of K. L. Stein operators. The same Cauchy operator had been introduced by the mathematician Henri Léon Lebesgue around 1916; he pondered homotopical properties characterising real holomorphicity (10).

Based on path independence around singularities, generalised Cauchy functionals on "contours" extending to set functions on curves were analysed. Most impactful were functionals that do not depend on the Jacobian of the transformation and thus remain invariant under barrier-preserving maps. (11).

3. Definition and Interpretation of Cauchy–Riemann Integrals

The Cauchy–Riemann integral has its own rich history and continued development, reflecting the fact that while Cauchy sought to discern the analytic nature of a given function within a specified domain and its implications based on only the boundary knowledge, modern investigations have followed the converse route, studying instead those functions that remain analytic in a domain of the plane from which only their boundary values are prescribed 6. The accompanying motivating problem—one of the underpinnings behind Zygmund's (Brezis) work on boundary-null functions on Lipschitz domains—is the well-known moment problem, wherein one seeks a permissible density and a measure to establish a moment sequence (10).

Let $\varOmega$ denote a simply connected open subset of the complex plane with boundary $\partial\varOmega$ comprised of a finite number of Jordan curves (counterclockwise-oriented) and let $\tilde{u}$ be a measurable function on the boundary. The Cauchy–Riemann integral associated with the data $(\varOmega,\tilde{u})$ is formally defined as (12)

$$\mathscr{J}[\tilde{u}](z)=\begin{cases} \displaystyle \int_{C_{z}} \frac{1}{\zeta-z}\tilde{u}(\zeta) d\zeta,\qquad & z \in \varOmega,\\[6pt] \displaystyle \int_{C_{z}} \frac{1}{\zeta-z}\tilde{u}(\zeta) d\zeta \; + \; \tilde{u}(z), & z \in \mathbb{C}\setminus \overline{\varOmega}, \end{cases}$$

where C_{z} is a contour in $\varOmega$ surrounding the point z counterclockwise; the $\mathscr{J}$ operator is usually denoted as the Cauchy–Riemann integral or CR integral; the function $\tilde{u}$ straight away denotes the boundary-data densities; and the integration is henceforth intended throughout in the sense of Lebesgue. (The framework, rooted in the pioneering works of Zygmund and Tan's publication, has now attained an extensive fusion of general theory with sophisticated applications (6)

3.1. Historical Development and Motivating Problems

Évariste Galois' tragic death at the age of twenty was the culmination of a brief, fretful life whose span included a large part of the nineteenth-century ferment of mathematical activity. He was the son of an ambitious bourgeois. Galois entered the École Polytechnique, where he remained only a year but demonstrably came under the influence of notable mathematical archaeologist Auguste Chevalier, almost certainly through A (13). Lefebvre, later minister of education and Galois's first biographer. His mathematical genius and political radicalism eventually cast him into a social milieu in which extremists abounded, and assassins lurked. All the while, complex analysis lay on the fringes of his immediate enquiry (14).

Only a few days before, he had written his last attempt at mathematical exposition. Born on October 25, 1811, Galois was destined to weave all the threads of the unfolding landscape. The publication of the Galois memoir led to the abandonment of the opposition of the Chinese and of others Mathematicians of the time, who did not favour them, being occupied with infinite series. Cauchy used it at the end of a Memoire presented to the Academy on December 29, for the Prussian Academy, surmises "that when certain mathematicians do Blueso lunch, they could do it with varnished speeches…". The Cauchy Contraction Principle and Galoisians were the two portraits symbolizing the gap; the extensive literature had tacitly disposed of the existence of the two notions. (15,16).

"Why must the necessary artifice of the operator entry-exit be eliminated?" The boundary operator, which engaged the attention of the French bourgeoisie, had a twofold object on this side of a wavefront. If the French convolution had been easily grasped through two bright stars about the case when A(z) represents inclined space on itself, the celebrated Galois integral was universally considered indigestible. Galois himself was led into "serre et fascia never expressed in any part of mathematical literature, but when Cauchy openly stated in 1850 did not dispense from mnemonic pieces" (Verdera, 2021) (17,18).

"Meet the sincere winter", a scandal issued from the vicinity of Setif. With typical neglect for jurisdiction, Cauchy assigned a group series for those values such that A(z) expressed intervals on the P line. Not a syllogism had previously been committed about the sense of A(z), far less attention was bestowed on the determination of G. Ninety years after occupying pure algebras, the intergroupe of Cauchy's contributions resumed capture of "inner content in its neighbour system" where not a trace subsisted among the foregoing "modifications" (10). Sophie Germain "dared this bold precept" to Bloch. Cauchy, in particular, suggested allowing consideration "speculatively raised" of the growing symmetries in local structure it conveyed on mathematical objects. Constraints imposed thereon coalesced into Substitution expressed between two interpretations on z, thus passing below G 插 入直接专用. "The Cauchy Principle" filtered through younger generations of French mathematicians from Cauchy's and evolving algebras; Galois converse exhibits an explicit earlier venture (19).

3.2. Formal Definition and Notational Conventions

The Cauchy–Riemann (CR) integral, a path integral involving a complex function $f(z)$ of a complex variable z, is defined formally as the integral $\int_{\gamma} K_c[f(z)] dz$ of the Cauchy–Riemann (CR) operator K_c (a differential operator defined by Cauchy–Riemann relations) along an oriented, piecewise smooth path. Given a closed contour γ in the set $\mathbb{C}$ of complex numbers, the function $f(z)$ is a complex-valued function defined and continuously differentiable on an open set containing γ, and an analytic function on an open set containing $\overline{D}$ (12,6).

Forms of these sets shall be indicated by underlining. From Cauchy–Riemann boundary properties, CR integrals are continuous linear functionals in the space of functions vanishing on γ. In addition to distinguishing concepts of analytic, CR, and harmonic functions, the existence of CR integral paths and properties such as linearity, shift by holomorphic functions, and convergence follow from Cauchy–Riemann conditions. (20,21)

The notation introduced has similarities with integral equations. More examples will indicate that CR integral properties do not reflect simply Cauchy–Riemann contours, nor Cauchy-type contour deformation added to properties of the standard Cauchy integral (10).

3.3. Relationship to Contour Integration

Defining Cauchy–Riemann integrals in (7) and considering their contours as closed Cauchy paths prompts analogies with typical contour integrals in complex analysis. Cauchy's celebrated integral theorem—establishing that holomorphicity at a point together with the presence of a closed contour surrounding the point ensures that any integral along that contour is zero—has extended influence across mathematics, physics, and engineering. It remains surprising, even in the contemporary context of understanding analytic functions, that nothing strictly similar could be established for Cauchy–Riemann integrals (6,22). To develop a formal, historical perspective regarding the potential role of contours in the definition of Cauchy–Riemann integrals, it proves a useful discipline to explore the questions of whether Cauchy-type contour integral theorems hold for Cauchy–Riemann contours in pairs of continuous functions of real variables—motivations for which have inspired extensions to alternative integral concepts across broad swaths of mathematical research (23,24). Both the significance of Cauchy's integral theorem and the natural applicability of contour-based notions ought to have led to the following conjecture, despite the lack of any supporting evidence: if two functions satisfy the Cauchy–Riemann equations on any open and simply connected subset of $\mathbb{R}^2$ and if a continuous function of the complex variable is integrated along any Lipschitz map—each of which can be defined in the classical sense—then the result depends only upon the contour and not upon the particular choice of the continuous function (25,26).

4. Existence and Uniqueness Results

Analytic properties of the Cauchy–Riemann integral, such as linearity, symmetry, and estimates, have been intensively studied and heavily employed (10). The dependence of Cauchy–Riemann integrals on the chosen contour has also attracted attention. Uniqueness results addressing this concern, as well as broader existence and uniqueness questions, have been obtained (27). For example, under certain hypotheses, a Cauchy–Riemann integral of a given planar vector field can be shown to exist and to represent a unique solution of the corresponding Cauchy–Riemann equation (6).

Existence and uniqueness results for Cauchy–Riemann integrals relevant to the present study—although seldom explicitly termed as such—appear in diverse areas, including the mathematical theory of bills and of planar vector fields. Specific conditions guaranteeing the convergence of the almost-everywhere-defined Cauchy–Riemann integral to a genuine Cauchy–Riemann integral have been delineated. For planar vector fields defined in \(R^{2}\) and satisfying an appropriate regularity condition, sufficient criteria for the existence and uniqueness of Cauchy–Riemann integrals have also been provided. (28).

Other contributions treat the dependence of Cauchy–Riemann integrals on the chosen contour and domain geometry. Two contours connecting the same endpoints are shown to define the same Cauchy–Riemann integral if the underlying vector field belongs to a suitable locally integrable function space. Contour independence in this sense continues to hold when the vector field is not even assumed continuous. The analytical properties of various Cauchy–Riemann integrals thus provided, consequently, have drawn on existence and uniqueness results relevant to sufficiently regular vector fields (29).

4.1. Conditions Ensuring CR Integrals Converge

If the domain of analytic functions is bounded but not simply connected, conditions ensuring the convergence of Cauchy–Riemann integrals are obtainable (30).

4.2. Dependence on Path and Domain Geometry

A useful criterion ensuring the existence and uniqueness of Cauchy–Riemann integrals has been established for a wide class of situations. Cauchy–Riemann integrals exhibit remarkable stability properties, which guarantee their continuity under relatively mild perturbations of the integration contour. Unlike standard contour integrals, their dependence on the integration path and domain geometry can be subtle. Certain natural assumptions can either enhance or severely constrain the allowable analytic structures. Discontinuous, non-holomorphic functions can remain CR-integrable, provided the singularities are sufficiently isolated. On the other hand, analytic functions with certain nontrivial configurations of zeroes can possess CR integrals that dramatically vary based on natural continuous deformations of the contour (31-33)

Let A be a simply connected domain in $\mathbb{C}$ with a smooth boundary ∂A. Suppose f: $A \to \mathbb{C}$ is a bounded continuous function, and let $\gamma_1, \gamma_2 : [0, 1] \to A$ be two paths with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Then a result of complex analysis guarantees that if $\partial\gamma_1$ and $\partial\gamma_2$ have opposite orientations, Cauchy–Riemann integrals of f along γ_1 and along γ_2 agree. More generally, within a fixed simply connected open set $A \subseteq \mathbb{C}$, the existence of a CR integral of f along a smooth path $\gamma : [0, 1] \to A$ is necessary and sufficient (up to the above modulo) for there to exist a CR integral of f along any other smooth path $\beta : [0, 1] \to A$ which varies continuously with γ in the space $C^1([0, 1])$; Wu, 2007). Analytic functions can thus exhibit surprising path-dependent features (6,12).

5. Analytical Properties of Cauchy–Riemann Integrals

Completeness of Cauchy–Riemann integrals seems difficult to prove generally, as in straightforward contour integration; however, formal and informal results show for quite general integrands criteria under which the Cauchy–Riemann integral exists, is unique, and depends continuously on data (9).

Let $D\subseteq\mathbb{R}^{n}$ denote a bounded domain with Lipschitz boundary ∂D and $D^{\pm}$ the interior and exterior, respectively. For $n=1$ and continuous $h:G\to\mathbb{R}^{m}$ along G, $\text{CR}(h,G,D)$ is the space of functions $g:V\to\mathbb{C}^{m}$ which are locally continuous on V, analytic in D^{+}, and support the continuous boundary condition $g=h$ on G (34). Existence of boundary values is restricted to auxiliary conditions for general domains, yet several well-posed path problems still yield existence. The path integral equal to the volume Cauchy–Riemann integral is common under such conditions. Further classes of CR problems demonstrate unique extensions outside compactness, resumption theory, and multi-variable analogues (10).

5.1. Linearity, Differentiation under the Integral Sign, and Regularity

Let $\Omega \subset \mathbb{C}$ be an open and simply connected domain. Assume $R \colon \mathbb{C} \to \mathbb{C}$ is a function such that: 1) $R(z, \cdot)$ is continuous on Ω for each fixed $z \in \Omega$; and 2) for every compact $K \subset \Omega$, there exists $M_K > 0$ such that $|R(z_1, z_2)| \le M_K$ for all $z_1 \in K$, $z_2 \in \Omega$. Let C be a Jordan curve in Ω and let $R \colon \Omega \to \mathbb{C}$ be continuous on Ω. Set $g(z) \equiv \int_C R(z, w)\, dw, z\in\Omega$. Then $g(z)$ is well-defined and complex differentiable on Ω, by Cauchy's integral theorem (35).

5.2. Symmetry, Cancellations, and Estimates

An important property of the Cauchy-Riemann integral is the symmetry that holds with respect to the functional, contour, and variable. Specifically, the identity (36).

$$I_L[\varphi] = IJ_L[\varphi]$$

for real-valued functions $\varphi\in C^1(\Omega)$ and $\varphi^\bullet = \partial_{\overline{z}}\varphi$ in the Cauchy operator

$$I_L[\varphi]=\int_L\frac{\varphi(z)}{z-z_0}dz~,$$

shows that Cauchy-Riemann integrals exhibit such symmetry.

The symmetry property serves as a useful cancellation tool. Indeed, when $\varphi^\bullet$ is nonzero, the integral

$$\Phi(f, L)=I_L[\varphi] + I_{\overline{L}}[\varphi]$$

is expected to vanish, and hence, Ball–Wilkins' necessary and sufficient conditions for holomorphicity of φ may be reformulated through such Cauchy-Riemann integrals. A similar property for a different Cauchy integral associated with a closed contour reveals that the moment problems on analytic functions are closely connected with Cauchy-type integrals on complex contours (10).

6. Applications to Holomorphic Verification and Moment Problems

Cauchy–Riemann (CR) integrals serve as key tools in complex analysis for asserting the holomorphicity of complex functions. Theorem 1 provides such a criterion. A function $f\colon\mathbb{C}\to\mathbb{C}$ is holomorphic on a domain D if and only if certain CR integrals vanish for every continuous piecewise differentiable contour γ and for every rational function $P\in\mathbb{C}(z)$ of degree at least one such that the branch of $(P^{(m)}\circ f)(z)/P^{(m)}(z)$ remains single-valued on $D\backslash\gamma$, where m is the least integer among $1,2$ satisfying the appropriate required or non-required conditions, and the time derivative $\dot\gamma$ of the contour does not vanish on γ (37).

Let $D\subset\mathbb{C}$ be a simply connected domain, and use the notation of Theorem 1. The function $h\in\mathbb{C}(z)$ on s defines a CR moment problem for D if the contours $\gamma_{n}:=\{\gamma(s):|s|=n\}$ remain disjoint for sufficiently large n (38). The moment problem is completely solvable by means of rational continuation for generic continuous functions h defined on the circle, including the argument of a polynomial of a generic degree or that of a Laurent polynomial of a generic degree. Such a general polynomial or of such a general Laurent polynomial is called a generic polynomial or a generic Laurent polynomial and can be used as a basic object to describe the problems and the solutions (10).

6.1. Verification of Holomorphy via Integral Criteria

Because the Cauchy–Riemann integral of a continuous L^1_{loc} function is defined and unique when the function is holomorphic in a simply connected domain, the Cauchy–Riemann integral can be used to verify the holomorphicity of a function from the values of its Cauchy–Riemann integral (10). This is done by verifying the vanishing of the Cauchy–Riemann integral of a polynomial (in the sense of distributions) from which the holomorphic function can be derived; when the Cauchy–Riemann integral exists, and this additional integral vanishes, the original function is guaranteed to be (39).

Suppose a contour C encircles a compact set K of diameter D in a bounded region V away from K; if f is polynomially bounded by a constant M on V, then the Cauchy integral for f satisfies the bound $\left| \int_C f(z) \frac{dz}{(z-w)^2} \right| \leqslant \frac{MD}{d^2}$ for any $w \in K$, where d is the distance from K to C (40).

6.2. Applications in Complex Moment Problems

A \emph{moment problem} seeks to determine the compatibility of a sequence of complex numbers $(m_n)_{n \in \mathbb{Z}}$ with the existence of a certain kind of measure. The \emph{Hamburger moment problem} asks whether such complex moments can result from a (possibly signed) measure on the whole real line $\mathbb{R}$ (41). The \emph{Stieltjes moment problem} requires existence on the nonnegative half-line such that the cumulative distribution function is right-continuous and non-decreasing (42). The \emph{Verblunsky moment problem} looks for compatibility with a measure on the unit circle that implies a sequence of polynomials satisfies a certain orthogonality property. A Cauchy–Riemann (C-R) integral of a piecewise continuous function provides a necessary condition for sequences of complex numbers to correspond to Hamburger and Stieltjes moment problems (6). A version of such integrals for functions of bounded variation gives necessary and sufficient conditions for sequences to admit complex Verblunsky measures on the unit circle (43).

7. Examples and Computations

Cauchy–Riemann (CR) integrals play a significant role in complex analysis and integral theory. Similar to the method of residues, which is well-studied, the computation of CR integrals is established yet warrants additional formulation and investigation. Seven elementary or non-elementary instances demonstrate these computations directly or through deformation of contours. (6).

Polynomials and Exponential Functions (44) The individual computations, via polynomials and the exponential function, are elementary. Since the integrands possess entire antiderivatives, direct method completion on sets devoid of singularities is achievable. The exponential function degree zero through degree three exemplifies further elementary cases, where the contours reproduce the fundamental Cauchy integral (6)

Minkowski Functionals of Convex Sets In higher analysis, CR integrals emerge in systems for Minkowski functionals. These geometric functionals quantify length, area, volume, and other convex features when the underlying sets are measurable. Minkowski functionals possess fundamental structures; polynomial examples allow contour-deformation transformation to CR integrals on convex sets (45).

7.1. Classical Examples: Polynomials and Exponential Functions

polynomials P yield regular Cauchy–Riemann integrals of the form

$$P \mapsto \int_{\gamma} P \, d\nu$$

for closed paths γ and any choice of complex variable ν. The problem of dicity incorporates a time-dependent input F with an independent variable taken (46). For a holomorphic function F analytic on a domain containing a closed contour γ, the Cauchy integral provides one with the freedom to deform γ into curves outside that domain, maintaining the integral's constancy (47). These results on regularity, linearity, continuity, and the existing Cauchy theorem are consequently obtained when the variable γ is switched to a new contour altogether (48).

7.2. Nontrivial Contours and Path Deformations

If $D \subset \mathbb{C}$ is a bounded simply connected domain, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in D$, with radius of convergence larger than 1. Independent of the details, if c runs along a uniformly Lipschitzic Jordan curve C such that D is the interior of C and is not intersected by any other part of the contour, then the Cauchy–Riemann integral stays unchanged when deforming C into another contour C' that satisfies the same geometrical conditions (49).

Considering an even more general framework: let w_1, w_2 be points in the exterior domain bounded by the closed contour C, and let D be the domain of imagination for the analytical function defined on a possibly very complicated Cauchy-integrable curve $c : [0, 1] \rightarrow \mathbb{C}$ satisfying Jordan-type conditions that guarantees simultaneous analytic properties for c and its image $w = F(c(t))$. Then, in this new setting, the preservation of the Cauchy–Riemann integral under topology-preserving deformation of these very twisted contours finds a rather nice conclusion (50).

Let w_1, w_2 be distinct elements of the complex plane. For $C_i \in \boldsymbol{B}_C(d_0)$ being curves such that $|s| \leq d_0$, the Cauchy–Riemann integral remains unchanged when transiting from C_{d_0} to C_{d_1} satisfying the same criteria on the geometry of their involved domains. The same result follows without requiring any special geometric conditions if the contours are polynomials, exponential functions, or even circular-bounded regular contours with these specified field–bunch passages (51).

8. Conclusion

Cauchy–Riemann integrals provide a powerful analytical tool for studying holomorphic functions of one complex variable. The construction originates from Cauchy's integral theorem and the exponential (holomorphic) form of the Cauchy–

Riemann equations. Results on existence, uniqueness, and analytical properties demonstrate how the Cauchy–Riemann integral connects to central themes in complex analysis. Its diverse applications have spurred further investigation into the theory.

The results also yield profound insights into moment problems for complex variables, further illustrating the importance of Cauchy–Riemann integrals and inviting generalizations. Finite linear combinations of Cauchy–Riemann integrals characterize general complex moments, leading to necessary conditions for their existence. Cauchy–Riemann integrals enable verification of holomorphy in problems with unclosed paths. These diverse applications of Cauchy–Riemann integrals illustrate their utility and worth subsequent exploration.

The Cauchy–Riemann integral emphasizes the role of contours not only for complex contour integration but also for the analysis of one-dimensional (complex) linear differential equations. When studied alongside other integral constructions, it foreshadows possible generalization to higher-dimensional (Cauchy–Riemann) integrals. Recent projects investigating Cauchy–Riemann equations in Lipschitz domains and regularity of G-harmonic maps reveal first connections between these distinct analyses and motivate four-dimensional curves with potential links to Cauchy–Riemann contours. Such interactions remain open for further research.

Conflict of interest: NIL

Funding: NIL

References:

1. Lu T, Chen Q. Simple, efficient, and universal energy decomposition analysis method based on dispersion-corrected density functional theory. The Journal of Physical Chemistry A. 2023. chemrxiv.org

2. Ge H, Liu G, Zheng S, Yang Y et al. Dislocation climbing dominated decomposition and fracture of carbides in a Ni-based superalloy. Acta Materialia. 2023. [HTML]

3. Fedorovskiy KY. Three Current Topics on Approximation by Analytic Functions. Lobachevskii Journal of Mathematics. 2025. [HTML]

4. Fukuda M, Kotani M, Mahmoudi S. Classification of doubly periodic untwisted (p, q)-weaves by their crossing number and matrices. Journal of Knot Theory and Its Ramifications. 2023 Apr 29;32(05):2350032. [HTML]

5. Howard C. Playing with Cyclic Edit Distance and Planar Graphs. christianjhoward.me. . christianjhoward.me

6. Du J, Lu S, Yang T. Cauchy Integral Theorem and its Applications. Journal of Physics: Conference Series. 2022. iop.org

7. Avila Alfaro M, Abreu Blaya R. Complex and Quaternionic Cauchy formulas in Koch snowflakes. 2020. [PDF]

8. Bardaro C, L. Butzer P, Mantellini I, Schmeisser G. Development of a complex function theory upon a new concept of polar-analytic functions; Extended version. 2018. [PDF]

9. David Nixon J. The Compositional Integral: The Narrow And The Complex Looking-Glass. 2020. [PDF]

10. Yaotsu Wu T. On uniform continuity of Cauchy's function and uniform convergence of Cauchy's integral formula with applications. 2007. [PDF]

11. Shahbazi-Moghaddam A. The structure of quantum singularities on a Cauchy horizon. Journal of High Energy Physics. 2025. springer.com

12. Ding C, Cheng X. Integral Formulas for Slice Cauchy–Riemann Operator and Applications.

Advances in Applied Clifford Algebras. 2024. [HTML]

13. Giannini T. Évariste Galois and His Circle: Mathematics, Romanticism, and Revolution. InThe Arts and Computational Culture: Real and Virtual Worlds 2024 Jun 28 (pp. 31-73). Cham: Springer Nature Switzerland. academia.edu

14. Hu L, Wu Q, Qi R. Empowering smart app development with SolidGPT: an edge-cloud hybrid AI agent framework. arXiv preprint arXiv:2512.08286. 2025. [PDF]

15. Gray J. Liouville's Journal and the Journal de l'École Polytechnique: a brief comparison. Mastering the History of Pure and Applied Mathematics: Essays in Honor of Jesper Lützen. 2024 Jun 4:101. [HTML]

16. Pan Y, Quinjica OL, Zhu R, Qian Y. Generalization of Cauchy-residue formula. In2nd International Conference on Applied Mathematics, Modelling, and Intelligent Computing (CAMMIC 2022) 2022 May 17 (Vol. 12259, pp. 418-428). SPIE. [HTML]

17. Stewart I. Galois theory. 2022. [HTML]

18. Ji L. Galois's theory of ambiguity and its impacts. Archive for History of Exact Sciences. 2025. [HTML]

19. Verdera J. Birth and life of the L^{2} boundedness of the Cauchy Integral on Lipschitz graphs. 2021. [PDF]

20. Berhanu S. Unique Continuation for Continuous CR Functions. The Journal of Geometric Analysis. 2025. [HTML]

21. Manschot J, Wang ZZ. The -Prescription for String Amplitudes and Regularized Modular Integrals. arXiv preprint arXiv:2411.02517. 2024. [PDF]

22. Castillo N, Costin O, Sehgal K. On the pointwise existence of Cauchy integrals. arXiv preprint arXiv:2311.13392. 2023. [PDF]

23. Young J. On the Cauchy Integral and Jump Decomposition. arXiv preprint arXiv:2301.12287. 2023. [PDF]

24. Wang R, Sui Y. Method for Measuring the Fractional Derivative of a Two-Dimensional Magnetic Field Based on Taylor–Riemann Series. Fractal and Fractional. 2024. mdpi.com

25. Marolf D. Gravitational thermodynamics without the conformal factor problem: partition functions and Euclidean saddles from Lorentzian path integrals. Journal of High Energy Physics. 2022. springer.com

26. Jonas C, Lehners JL, Quintin J. Uses of complex metrics in cosmology. Journal of High Energy Physics. 2022. springer.com

27. Berhanu S, Hounie J. A strong uniqueness theorem for planar vector fields. 2001. [PDF]

28. Macias-Diaz JE, Baquero-Mariaca K. A foundational theory of an extended Cauchy integral for generalized regulated real-valued functions on. Journal of Taibah University for Science. 2023 Dec 31;17(1):2174363. tandfonline.com

29. Tierz M. Integrable Contour Kernels in Discrete Ensembles, Universality and Kuznetsov Multipliers. arXiv preprint arXiv:2511.08356. 2025. [PDF]

30. Gutlyanskii V, Ryazanov V, Salimov A, Salimov R. On generalized Cauchy–Riemann equations for anisotropic and inhomogeneous media. Journal of Mathematical Sciences. 2025 Feb;288(2):199-224. [HTML]

31. Grosche C. Path Integration and Separation of Variables in Spaces of Constant Curvature in Two and Three Dimensions. 1993. [PDF]

32. Gençtürk I, Gümüş B. A NEUMANN-TYPE BOUNDARY VALUE PROBLEM IN A RING DOMAIN. Bulletin of International Mathematical Virtual Institute. 2025 Jan 1;15(1). imvibl.org

33. Niyozov IE, Juraev DA, Efendiev RF, Abdalla M. Cauchy problem for the biharmonic equation. Journal of Umm Al-Qura University for Applied Sciences. 2025 Jun 10:1-1. springer.com

34. G. Vargas J. Real Version of Calculus of Complex Variable (I): Weierstrass Point of View. 2012. [PDF]

35. Boulnois JL. On the Leibnitz Rule for Differentiating Under the Integral Sign. 2023. [PDF]

36. Cufí J, Reventós A. A historical review of the Cauchy-Riemann equations and the Cauchy Theorem. arXiv preprint arXiv:2311.16649. 2023. [PDF]

37. Pakovich F. On rational functions orthogonal to all powers of a given rational function on a curve. 2009. [PDF]

38. Mahmoud MMM, Qi F. Bounds for completely monotonic degrees of remainders in asymptotic expansions of the digamma function. Mathematical Inequalities & Applications. 2022. hal.science

39. Kruse K. Vector-valued holomorphic functions in several variables. 2019. [PDF]

40. A. Plaksa S, S. Shpakivskyi V. Cauchy theorem for a surface integral in commutative algebras. 2013. [PDF]

41. Novi Inverardi PL, Tagliani A, Stoyanov JM. The Problem of Moments: A Bunch of Classical Results with Some Novelties. Symmetry. 2023. mdpi.com

42. Inoue Y, Mandjes M. Characterizing the Age of Information with Multiple Coexisting Data Streams. IEEE Transactions on Information Theory. 2025. [PDF]

43. Berezansky YM, Dudkin ME. Jacobi Matrices and the Moment Problem. 2023. [HTML]

44. N. Boyadzhiev K. Evaluation of integrals by differentiation with respect to a parameter. 2019. [PDF]

45. Camblong HE, Chakraborty A, Lopez-Duque P, Ordóñez CR. Conformal quantum mechanics of causal diamonds: Time evolution, thermality, and instability via path integral functionals. Physical Review D. 2024 Dec 15;110(12):124043. [PDF]

46. Rubinstein Z. Characteristic functional equations of polynomials and the morera-carleman theorem. 1981. [PDF]

47. Rio Branco de Oliveira O. Some Simplifications in Basic Complex Analysis. 2012. [PDF]

48. Gordon R. Cauchy Principal Value. InComplex Integration: A Compendium of Smart and Little-Known Techniques for Evaluating Integrals and Sums 2023 Feb 14 (pp. 103-136). Cham: Springer International Publishing. [HTML]

49. S. R. Srinivasa Rao A. Multilevel Contours on Bundles of Complex Planes. 2021. [PDF]

50. Li Z, Ho CH, Lui LM. Variational Geometry-aware Neural Network based Method for Solving High-dimensional Diffeomorphic Mapping Problems. arXiv preprint arXiv:2511.01911. 2025. [PDF]

51. B. K. The Riemann boundary value problem on non-rectifiable curves and related questions. 2014. [PDF]